The Torelli group

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1. The mapping class group of a surface Mod (S)

The **mapping class group** of a surface S is the group

$$\text{Mod}(S) = {}^{\text{Homeo}^+(S, \partial S)}\!/\!{}_{\sim},$$

where $\sim$ denotes smooth isotopy relative to the boundary.

A **mapping class** in Mod (S):

- is an isotopy class of orientation-preserving homeomorphism of S,
- preserves the marked points of S setwise,
- preserves ∂S pointwise.

2. Dehn twists in Mod (S)

Parameterise a regular annular neighbourhood A of a simple closed curve c on a surface S by $A = re^{i\theta}$. The **twist map about c** is the homeomorphism $(r, \theta) \mapsto (r, \theta + 2\pi r)$.

A **Dehn twist about c**, denoted T_c, is the homeomorphism induced by performing the twist map about c and extending by the identity on the rest of S.

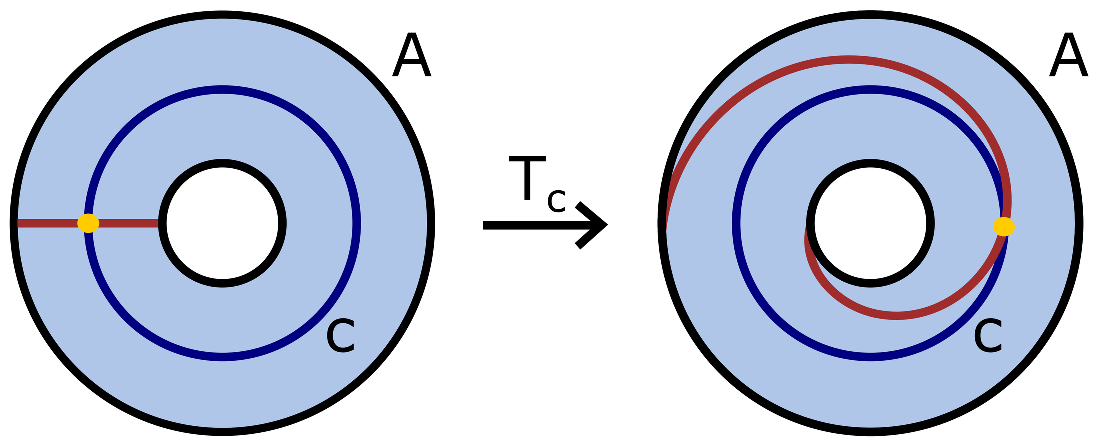

Figure: A Dehn twist T_c.

The Dehn twist T_c is an important mapping class in Mod (S):

- T_c is orientation preserving,
- ∂A is fixed pointwise under T_c, so ∂S is fixed pointwise,
- marked points in S are preserved setwise,
- T_c is well-defined on the isotopy class of c.

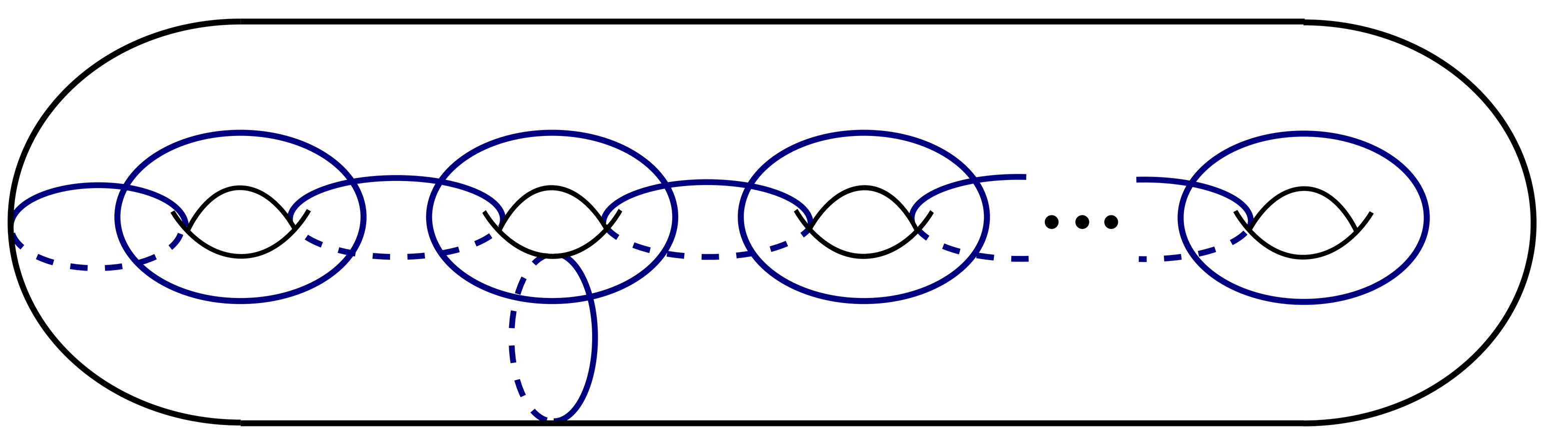

Figure: The mapping class group of a genus g surface Mod (S_g) is generated by $2g + 1$ Dehn twists, known as the Humphries generators.

3. Intersection numbers

Consider two oriented simple closed curves a and b on an oriented surface S_g.

- **Geometric intersection number:**
$$i(a, b) = \min\{|a \cap b| \;:\; a \in [a]\,,\; b \in [b]\}.$$
- **Algebraic intersection number:**
$$\hat{i}(a, b) = \sum_{p \in a \cap b} \text{sgn}(p),$$
where sgn(p) $= 1$ if the orientation of the intersection point p agrees with the orientation of S_g and sgn(p) $= -1$ otherwise.

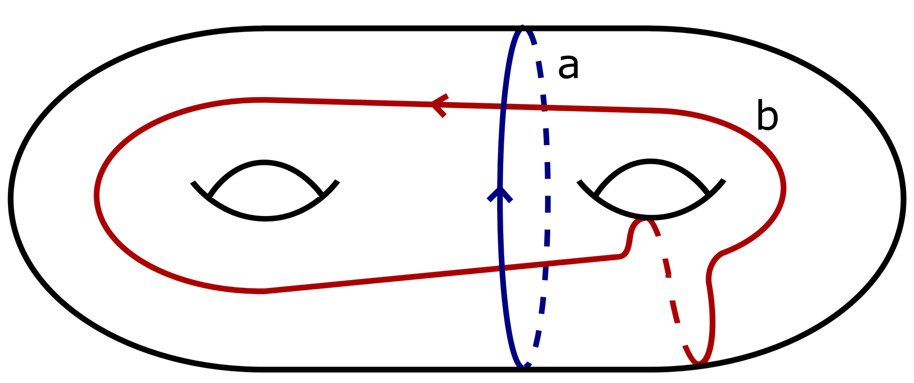

Figure: A pair of curves, a and b, with $i(a, b) = 2$ and $\hat{i}(a, b) = 0$.

The algebraic intersection number is a **symplectic form** on the first homology group of a genus g surface.

4. The Torelli group $\mathcal{I}(S_g)$

The **Torelli group** $\mathcal{I}(S_g)$ is the kernel of the symplectic representation ρ, expressed in the exact sequence:

$$1 \longrightarrow \mathcal{I}(S_g) \longrightarrow \text{Mod}(S_g) \xrightarrow{\rho} Sp_{2g}(\mathbb{Z}) \longrightarrow 1$$

where $Sp_{2g}(\mathbb{Z})$ denotes the integral symplectic group.

The symplectic representation ρ is generally not faithful - so it makes sense for us to talk about the Torelli group.

The action of a Dehn twist on homology is described by the formula:

$$\rho(T_b^k)([a]) = [a] + k \cdot \hat{i}(a, b)[b].$$

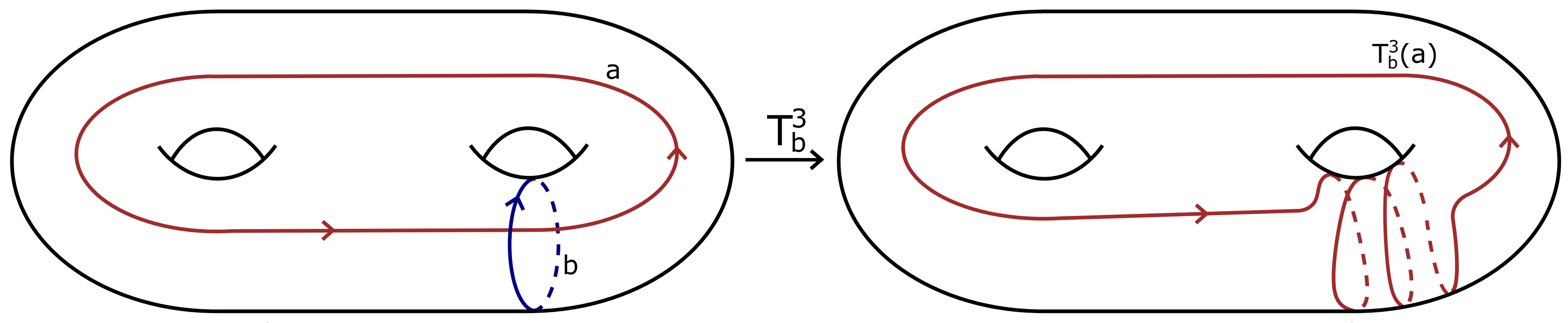

Figure: $T_b^3(a)$ represents the homology class obtained by applying T_b^3 to $[a]$.

Mapping classes in the Torelli group act trivially on homology:
If $x \in \mathcal{I}(S_g)$ then $\rho(x)([a]) = [a]$ for all $[a] \in H_1(S_g, \mathbb{Z})$.

5. Mapping classes in the Torelli group $\mathcal{I}(S_g)$

Let a be a separating simple closed curve on S_g.

- **Separating twists** T_a are in $\mathcal{I}(S_g)$.

Let b and c be non-separating oriented simple closed curves on S_g.

- **Bounding pair maps** $T_b T_c^{-1}$ are in $\mathcal{I}(S_g)$, where $[b] = [c]$, $\hat{i}(b, c) = 0$ and $i(b, c) = 0$.

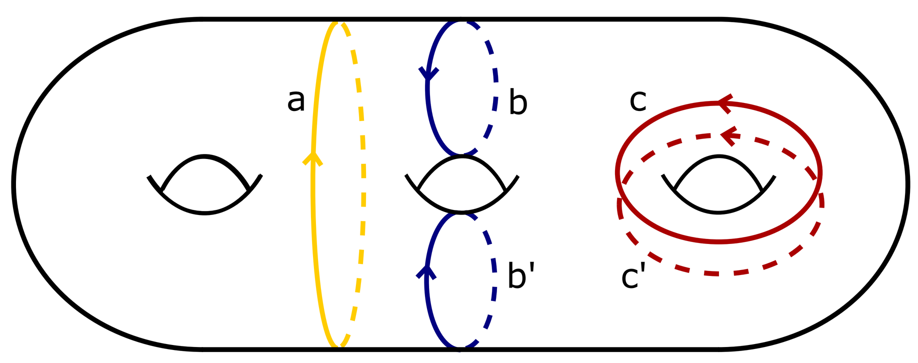

Figure: The curve a defines a separating curve, while pairs b and b', c and c' are examples of bounding pairs.

- **Fake bounding pair maps** $T_b T_c^{-1}$ are in $\mathcal{I}(S_g)$, where $[b] = [c]$, $\hat{i}(b, c) = 0$ and $i(b, c) \neq 0$.
- **Simply intersecting pair maps** $[T_b, T_c] = T_b T_{T_c(b)}^{-1}$ are in $\mathcal{I}(S_g)$, where $\hat{i}(b, c) = 0$ and $i(b, c) = 2$.

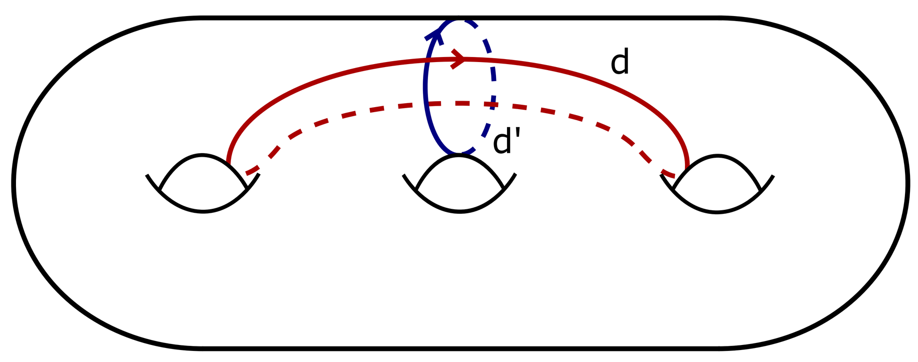

Figure: The pair d and d' define a special case of a fake bounding pair: a simply intersecting pair. Notice $[d] \neq [d']$, but that $[d] = [T_{d'}(d)]$.

6. Remarks on the Torelli group

- The Torelli group $\mathcal{I}(S_1)$ is trivial.
- The Torelli group can be generated by separating twists and bounding pair maps.
- The Torelli group plays a role in the study of other groups, such as in the representation theory of the braid group.

References

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[3] B. Farb and D. Margalit, *A primer on mapping class groups*, vol. 49, Princeton Mathematical Series, Princeton University Press, Princeton, NJ, 2012.

[4] J. Powell, "Two theorems on the mapping class group of a surface", *Proc. Amer. Math. Soc.* 68.3 (1978), 347–350.